Chirol Hoszul Duality

Last week: We saw fop. faet. algebras, a generalization of E2-algebras. Those were sheaves on the Ran Space. An Ez-algebra is a locally const. fact. algebra. In particular, we can take to disjoint disks & collide them, and get an actual multiplication. We than have an St-space of mult.

Today: Describe à polomorpline (de-Rham) version. The main Jeuture is having operation product expansion (OPE): Toke the Laurent expansion of  $\lim_{|x-y| \to 0} (a_x, b_y)$ . The 2-st formalism: \$1.5-VOA: a vector Space V equipped w/ a unit, a derivation &  $V \otimes V \longrightarrow V [z^{\pm 1}]$  $a \otimes b \longrightarrow \sum (a_n b) z^{-n-1}$ ر(بع)) ۳

sotisfying a long list of axioms. (a fact alg on a formal disk") Det by Borcherds (1986). \$1 Chiral Algebras A global reformulation by Bailingon & Drinfeld. Idea: X/K proper smooth curbe in char o  $A \in \mathcal{D}$ -mod(X). Let  $\mathfrak{f}: X^2 \backslash \Delta \hookrightarrow X^2 \mathfrak{c} \to X : \Delta$ A Chiral olge str on A is  $\mathcal{Y}^{:} \mathfrak{Z}_{\star} \mathfrak{Z}^{!} \mathfrak{A}^{\mathbb{Z}^{2}} \longrightarrow \Delta_{\star} \mathfrak{A}$ 

Satisfying anti-Comm & Jacobiidenfity.

Ex: • The writ de alg wy:  $J_{x}: J_{x}J = \omega_{x}[J_{x}] \xrightarrow{\mathbb{D}^{2}} J_{x}J = \omega_{x}2[J_{x}] \xrightarrow{\text{adjunction}} J_{x}J = \omega_{x}2[J_{x}] \xrightarrow{\mathbb{D}^{2}} J_{$  $\Delta_* \Delta^! \omega_{\chi^2} [-1] \xrightarrow{\sim} \Delta_* \omega_{\chi} [-1]$ · More generally, for A & CAlg (D-mod(X))  $\mu: \mathfrak{z}_{\star} \mathfrak{z}^{!} \mathfrak{A}^{\mathbb{Z}^{2}} \longrightarrow \Delta_{\star} \Delta^{!} \mathfrak{A}^{\mathbb{Z}^{2}} \simeq$  $\Delta_*(A \otimes A) \xrightarrow{m} \Delta_* A$ This is a Commutative Chiral Alg

Let: A Lie- \* alg is the same dafa og a Chiral olg except  $[-,-]: A^{\boxtimes^2} \longrightarrow \bigtriangleup_* A$ (who the polar part) Ex: Given a sheaf of Lie alg L, def d:=L∞Dx Prop: Universal enveloping U<sup>ch</sup>: Lie-\*(X) I ChAlg(X): obly  $\mathcal{U}^{ch}(\mathfrak{L})_{\chi} \cong \operatorname{Ind}_{H^{o}_{dR}(\mathfrak{D}_{\chi};\mathfrak{L})}^{H^{o}_{dR}(\mathfrak{D}_{\chi};\mathfrak{L})} \mathfrak{k}$ 

$$\begin{aligned} & \{ \chi_{\kappa} := \mathcal{U}^{ch}(g \otimes D_{\chi}) : \text{ Hore generally} \\ & \mathcal{A}_{g} := \mathcal{U}^{ch}(g \otimes D_{\chi}) : \text{ Hore generally} \\ & \mathcal{O} \Rightarrow \omega_{\chi} \Rightarrow \mathcal{I}_{g,\kappa} \Rightarrow g \otimes D_{\chi} \Rightarrow 0 \\ & \mathcal{N} \Rightarrow \mathcal{A}_{g,\kappa} := \mathcal{U}^{ch}(\mathcal{J}_{g,\kappa})/(4-1) \text{ is flow} \\ & \mathcal{H}_{ae} \cdot \mathcal{M}_{ody} \quad Chiral \quad alg \\ & (\mathcal{A}_{g,\kappa})_{\chi} \cong \mathcal{V}_{g,\kappa} : \\ & \mathfrak{I}_{\kappa} \mathfrak{I}^{!} \mathcal{A}_{g,\kappa}^{\mathfrak{B}_{2}} \longrightarrow \mathcal{D}_{\kappa} \mathcal{A}_{g,\kappa} \\ & \mathfrak{I}(\mathfrak{Z}_{1} \mathbb{N})(\chi_{-1} \boxtimes \mathcal{J}_{m}) = \text{ polan part of} \\ & f(\mathfrak{Z}_{w}) \sum_{n} \chi_{n} \mathcal{Y}_{m}(\mathfrak{Z} - w)^{-n-2} \\ & (\chi_{n} = \chi_{0} \mathfrak{e}^{n}; \chi \in g) \end{aligned}$$

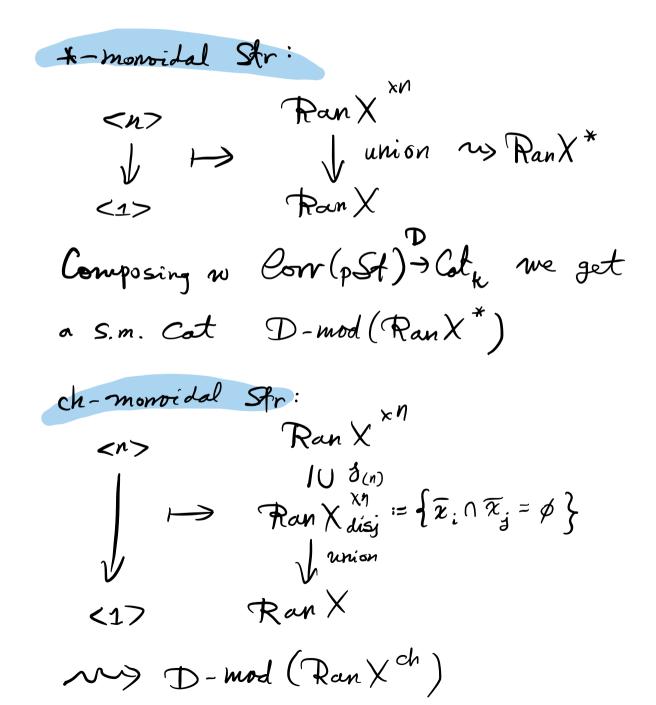
S2-Factori Sation Algebras  
We can also translate fact.  
alg to the tolomorphic setting:  
Let 
$$\operatorname{Ran} X := \operatorname{lin} X^{\mathrm{I}}$$
. Then  
 $\operatorname{fSat}^{\mathrm{Stop}}$ . Then  
 $\Delta_{(\alpha)}: X^{\mathrm{I}} \longrightarrow X^{\mathrm{J}} \longleftrightarrow \mathcal{U}_{(\alpha)}:= \{x_{j} \neq x_{i} : \alpha(i) \neq \alpha(j)\}: \mathfrak{Z}_{(\alpha)}$   
 $\mathcal{D}\operatorname{-mod}(\operatorname{Ran} X):= \operatorname{Ddim} \mathcal{D}\operatorname{-mod}(X^{\mathrm{I}})$   
 $\mathcal{M} \sim \{\mathcal{M}_{\mathrm{I}} \in \mathcal{D}\operatorname{-mod}(X^{\mathrm{I}}), \mathcal{M}_{\mathrm{I}} \xrightarrow{\sim} \Delta_{(\alpha)}^{\mathrm{I}} \mathcal{M}_{\mathrm{J}}\}$   
 $\mathcal{D}_{\mathrm{d}}: A \quad \text{factorization algebra}$ 

is BED-mod(RanX) + factorization structure  $J_{(\lambda)}^{!} \mathbb{B}_{\mathcal{J}} \xrightarrow{\sim} J_{(\lambda)}^{!} \boxtimes \mathbb{B}_{\mathcal{J}_{i}} +$ I compatibility.

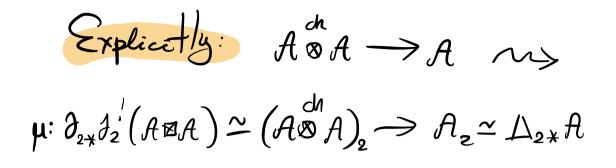
Given a fast alg &, define A = B1. We have a map  $\mu: \mathcal{J}_* \mathcal{J}^! \mathcal{B}_1^{\mathbb{Z}^2} \longrightarrow \mathcal{J}_* \mathcal{J}^! \mathcal{B}_2 \longrightarrow \mathcal{A}_* \mathcal{A}^! \mathcal{B}_2 \xrightarrow{\sim} \mathcal{A}_* \mathcal{B}_1$ factorization orgunation def of D-mod (Ran) Claim: (A,y) is a Chiral alg. & we have E: Fact Alg (X)-> ChAlg(X)

Thm (BD): I is an equivalence. Well see a proof by Francis-Gaitsgory. The main idea of the proof: Define a symm mon str. on D-mod (Ran X) s.t. : Lie (D-mod (RanX))~> co Alg (D-mod (RanX))  $ChAlg(x) \xrightarrow{\sim} FactAlg(x)$ \$3 - Monoidal Structures on Ranx Reminder: CAlg(@) S Fun(Fins, C)

Let pSt := pSh(Aff). We define 2 CAlg's in Corr(pSt):



Explicitly :  $\left(\bigotimes_{i\in\mathbb{T}}^{*}M_{i}\right)_{\mathcal{T}}\simeq\bigoplus_{\mathcal{J}\twoheadrightarrow\mathcal{I}}\boxtimes(M_{i})_{\mathcal{J}_{i}}$  $\begin{pmatrix} on \\ on \\ i \in I \end{pmatrix} \xrightarrow{\sim} \oplus \mathcal{J}_{(\alpha)} * \mathcal{J}_{(\alpha)} \stackrel{I}{\boxtimes} (M_i)_{J_i}$ Def/Prop: A chiral algebra (Lie-\* algobra) is an object AE Lie (D-mod (Ran X ch)) (resp. Ran X\*) s.t. the underlaying D-mod. is in the essential image of  $\Delta_{*}^{\text{main}}$ :  $D - mod(X) \rightarrow D - mod(BanX)$ 



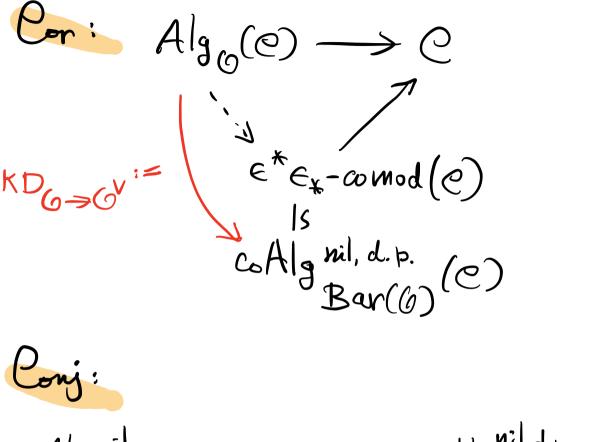
Ref / Prop: Let BE co CA/g (D-mod (Ran X ch ))  $\gg \mathcal{P}_{J} \rightarrow (\mathcal{P}^{\otimes I})_{T} \simeq$  $\sim j_{(a)}^{!} \mathbb{B}_{T} \rightarrow j_{(a)}^{!} \boxtimes_{\mathcal{J}_{i}} (x)$ We say R is a factori Zafion alg if (\*) is an equil for all d.

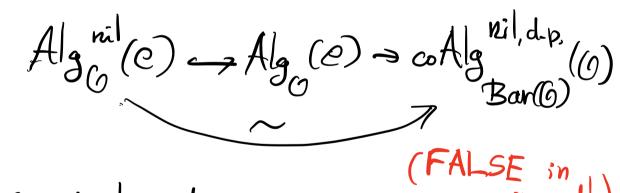
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Reminder: C monoidel Cat. Op(C) := Alg (C<sup>E</sup>), CoOp(C):= CoAlg (C<sup>E</sup>)  $Bar: Op(C) \longrightarrow coOp(C): coBar$ 0 H) ... ₹1.0.1 ₹1.1 \* **1** × 1 ″  $T_{t}(\dots \geq 1, P_{1} \geq 1, 1) \leftarrow P$  $T_{(-)}: Op(\mathcal{C}) \longrightarrow Monad(\mathcal{C})$  $e^{\varepsilon} \longrightarrow End(e)$  $S_{(-)}^{\dagger} \operatorname{colp}(\mathcal{C}) \longrightarrow \operatorname{coMonad}(\mathcal{C})$ 

 $\chi \longrightarrow \bigoplus_{n \ge 0} \left( \mathcal{O}(n) \otimes \chi^{\otimes n} \right)_{\Sigma_n}$  $M > Alg_{(0)} ( C ) := T_{6} - mod ( C )$ coAlg<sup>nil,d.p.</sup> (C) := So- comod (C) Remi coAlg (e) will corresp to  $\chi := \frac{1}{n \ge 0} \left( (O(n) \otimes \chi^{\otimes n})^{(\Sigma_n)} \right)$ Prop: Bar(Lie) ~ coComm [-1] Idea: 0 -> Lie -> Assoc -> Comm -> 0 So we want a map Alg (e) -> coAlg (e) Bar(6)

Let 
$$Alg_{\mathcal{G}}(\mathcal{O}) \xrightarrow{\epsilon_{*}} Alg_{1}(\mathcal{O}) \simeq \mathcal{O}$$
  
be the pull push along the augmentation.  
Lemma  $\epsilon_{*} \epsilon_{*} \simeq S \operatorname{Bar}(\mathcal{O})$   
I dea:  $S_{\operatorname{Bar}(\mathcal{O})} \simeq S_{1 \otimes 1}(\mathcal{X})$   
 $\simeq | \cdots \equiv 1 \otimes \mathcal{O} \otimes 1 \xrightarrow{\epsilon_{\otimes} \operatorname{id}} 1 \otimes 1 | (\mathcal{X})$   
 $\approx | \cdots \equiv \operatorname{id} \circ \mathcal{T}_{\mathcal{G}} \circ \operatorname{id}(\mathcal{X}) \xrightarrow{\epsilon_{\times} \operatorname{oid}} \operatorname{id}(\mathcal{X}) |$   
 $\operatorname{action commutes}$   
 $w| geo. realization$   
 $\approx | \cdots \equiv \mathcal{T}_{\mathcal{G}} \epsilon_{*} \times \rightrightarrows \epsilon_{*} \times |$   
 $\simeq 4 \otimes \epsilon_{*} \times \simeq \epsilon^{*} \epsilon_{*} \times$ 





Mill be four in our cose.

Upshot: D-mod (Ran X)~ colim D-mod (Ran<sup>≤n</sup>X)

Li Cym <sup>≤n</sup> (A[1])  $\underset{\mathcal{J} \to \mathcal{J}}{\overset{\mathcal{J}}{\to}} gr^{\mathsf{I}} \mathsf{CE}(\mathcal{A}) \simeq \operatorname{Sym}^{\mathsf{I}}(\mathcal{A}[1]) \simeq (\mathcal{A}^{\overset{\mathcal{J}}{\otimes}\mathsf{I}})_{\mathcal{E}_{1}}$   $gr^{\mathsf{I}} \mathsf{CE}(\mathcal{A})_{\mathcal{J}} \simeq (\bigoplus_{\alpha:\mathcal{J} \to \mathcal{J}} \mathfrak{I}_{(\alpha)} \times \mathfrak{I}_{(\alpha)}^{\mathsf{I}} \boxtimes \mathcal{A}_{\mathcal{J}_{1}})_{\mathcal{E}_{1}}$ The factorization property is ferr equily to  $J_{I}^{!}CE(A)_{I} \xrightarrow{\sim} J_{I}^{!}A[A]_{I}^{\boxtimes I}$  $\sim f_{I}^{!} \operatorname{gr}^{I} \operatorname{CE}(A)_{I}$  $: e \cdot j_i^T gr^T CE(A)^2 = 0 \quad |2| \neq |1|$ This happens preeisely if Ag is supported on a diagonal, So fot it will be annihilated

by  $j_{\pm}^{!}$ , which in turn is equiv. to  $A \simeq \Delta_{\star}^{\text{main}} A_{\pm}$ .