Chiral Koszul Duality

Last week: We saw top. fact. algebras, a generalization of $\mathcal{E}_2$-algebras. Those were sheaves on the Ran Space. An $\mathcal{E}_2$-algebra is a locally const. fact. algebra. In particular, we can take to disjoint disks & collide them, and get an actual multiplication. We then have an S^1-space of mult.

Today: Describe a holomorphic (de-Rham) version. The main feature is having operator product expansion (OPE):

Take the Laurent expansion of $\lim_{|x-y|\to 0}(a_x, b_y)$. The 1-st formalism:

§0.5-VOA:

a vector space V equipped w/ a unit, a derivation &

$$V\otimes V \to V[[z^{\pm 1}]]$$

$$a\otimes b \longmapsto \sum_n \underset{\overset{\uparrow}{\mathbb{C}((z))}}{(a_n b)} z^{-n-1}$$

satisfying a long list of axioms.

(a "fact alg on a formal disk")

Def by Borcherds (1986).

§1 Chiral Algebras

A global reformulation by Beilinson & Drinfeld. Idea:

$X/\bar{k}$ proper smooth curve in char 0

$A \in D\text{-mod}(X)$. Let

$$j : X^2 \setminus \Delta \hookrightarrow X^2 \hookleftarrow X : \Delta$$

A chiral alg str on A is

$$\mu : j_* j^! A^{\boxtimes 2} \longrightarrow \Delta_* A$$

satisfying anti-Comm & Jacobi-identity.

Ex:

- The unit ch alg ω_X:

$$\mu: j_* j^! \omega_X[-1]^{\boxtimes 2} \xrightarrow{\sim} j_* j^! \omega_{X^2}[-2] \xrightarrow{\text{adjunction}} \Delta_* \Delta^! \omega_{X^2}[-1] \xrightarrow{\sim} \Delta_* \omega_X[-1]$$

- More generally, for $A \in CAlg(D\text{-mod}(X))$

$$\mu: j_* j^! A^{\boxtimes 2} \to \Delta_* \Delta^! A^{\boxtimes 2} \simeq \Delta_*(A \overset{!}{\otimes} A) \xrightarrow{m} \Delta_* A$$

This is a Commutative Chiral Alg

Def: A Lie-$*$ alg is the same data as a Chiral alg except

$$[-,-] : A^{\boxtimes 2} \longrightarrow \Delta_* A$$

(w/o the polar part)

Ex: Given a sheaf of Lie alg L, def $\mathcal{L} := L \otimes \mathcal{D}_X$

Prop: Universal enveloping

$$\mathcal{U}^{ch} : \mathrm{Lie}\text{-}*(X) \overset{\perp}{\rightleftarrows} \mathrm{ChAlg}(X) : \mathrm{oblv}$$

$$\mathcal{U}^{ch}(\mathcal{L})_x \simeq \mathrm{Ind}_{H^0_{dR}(D_x;\mathcal{L})}^{H^0_{dR}(D_x^\times;\mathcal{L})} k$$

Ex: Let $\mathfrak{g}$ be a s.s. Lie alg.

$A_{\mathfrak{g}} := \mathcal{U}^{ch}(\mathfrak{g} \otimes \mathcal{D}_X)$. More generally

$$0 \to \omega_X \to \mathcal{L}_{\mathfrak{g},\kappa} \longrightarrow \mathfrak{g} \otimes \mathcal{D}_X \to 0$$

$\rightsquigarrow A_{\mathfrak{g},\kappa} := \mathcal{U}^{ch}(\mathcal{L}_{\mathfrak{g},\kappa})/(\mathbf{1}-1)$ is the Kac-Moody chiral alg

$(A_{\mathfrak{g},\kappa})_x \simeq \mathbb{V}_{\mathfrak{g},\kappa}$.

$$j_* j^! A_{\mathfrak{g},\kappa}^{\boxtimes 2} \longrightarrow \Delta_* A_{\mathfrak{g},\kappa}$$

$f(z,w)(x_{-1} \boxtimes y_m)$ = polar part of

$$f(z,w) \sum_n x_n y_m (z-w)^{-n-1}$$

$(x_n := x \otimes t^n ; x \in \mathfrak{g})$

§2 - Factorization Algebras

We can also translate fact. alg to the holomorphic setting:

Let $\operatorname{Ran} X := \lim\limits_{f\mathrm{Set}^{s,op}} X^I$. Then

$$\Delta_{(\alpha)} : X^I \longrightarrow X^J \hookleftarrow \mathcal{U}_{(\alpha)} := \{x_j \neq x_i : \alpha(i) \neq \alpha(j)\} : j_{(\alpha)}$$

$$\begin{array}{ccl} D\text{-mod}(\operatorname{Ran} X) := & \operatorname{colim} & D\text{-mod}(X^I) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathcal{M} & \sim & \{\mathcal{M}_I \in D\text{-mod}(X^I), \\ & & \mathcal{M}_I \xrightarrow{\sim} \Delta^!_{(\alpha)} \mathcal{M}_J\} \end{array}$$

Def: A factorization algebra

is $\mathcal{B} \in D\text{-mod}(\mathrm{Ran}\, X)$ +

factorization structure

$$j^!_{(\alpha)} B_J \xrightarrow{\sim} j^!_{(\alpha)} \underset{I}{\boxtimes} B_{J_i}$$ +

compatibility.

Given a fact alg $\mathcal{B}$, define $\mathcal{A} := \mathcal{B}_1$. We have a map

$$\mu: j_* \underbrace{j^! B_1^{\boxtimes 2}} \longrightarrow j_* \underbrace{j^! B_2} \xrightarrow[\text{adjunction}]{} \Delta_* \Delta^! B_2 \xrightarrow[\text{def of D-mod(Ran)}]{\sim} \Delta_* B_1$$

factorization

Claim: (A, μ) is a chiral alg.

& we have $\Phi: \mathrm{FactAlg}(X) \longrightarrow \mathrm{ChAlg}(X)$

Thm (BD): Φ is an equivalence.

We'll see a proof by Francis-Gaitsgory.

The main idea of the proof:

Define a symm mon str. on

$D\text{-mod}(\mathrm{Ran}X)$ s.t. :

$$\begin{tikzcd}
\mathrm{Lie}(D\text{-mod}(\mathrm{Ran}X)) \arrow[r, "\overset{CE}{\sim}"] & \mathrm{coCAlg}(D\text{-mod}(\mathrm{Ran}X)) \\
\mathrm{ChAlg}(X) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "\sim"] & \mathrm{FactAlg}(X) \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

§3 - Monoidal Structures on Ran X

Reminder: $\mathrm{CAlg}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{Fin}_*^s, \mathcal{C})$

Let $pSt := pSh(Aff)$. We define 2 CAlg's in $Corr(pSt)$:

$*$-monoidal Str:

$$\begin{tikzcd}
\langle n\rangle \arrow[d] & \mapsto & \operatorname{Ran}X^{\times n} \arrow[d, "\text{union}"] & \rightsquigarrow \operatorname{Ran}X^{*} \\
\langle 1\rangle & & \operatorname{Ran}X &
\end{tikzcd}$$

Composing w $Corr(pSt) \xrightarrow{D} Cat_k$ we get a s.m. Cat $D\text{-mod}(\operatorname{Ran}X^{*})$

ch-monoidal Str:

$$\begin{tikzcd}
\langle n\rangle \arrow[dd] & & \operatorname{Ran}X^{\times n} \\
 & \mapsto & \operatorname{Ran}X^{\times n}_{disj} := \{\overline{x}_i \cap \overline{x}_j = \phi\} \arrow[u, hook, "j_{(n)}"] \arrow[d, "\text{union}"] \\
\langle 1\rangle & & \operatorname{Ran}X
\end{tikzcd}$$

$\rightsquigarrow D\text{-mod}(\operatorname{Ran}X^{ch})$

Explicitly:

$$\left(\overset{*}{\bigotimes_{i\in I}} M_i\right)_J \simeq \bigoplus_{J\twoheadrightarrow I} \underset{I}{\boxtimes} (M_i)_{J_i}$$

$$\left(\overset{ch}{\bigotimes_{i\in I}} M_i\right)_J \simeq \bigoplus_{\alpha: J\twoheadrightarrow I} j_{(\alpha)*}\, j^{!}_{(\alpha)} \underset{I}{\boxtimes} (M_i)_{J_i}$$

Def/Prop: A chiral algebra (Lie-$*$ algebra) is an object $A \in \mathrm{Lie}(D\text{-mod}(\mathrm{Ran}\, X^{ch}))$ (resp. $\mathrm{Ran}\, X^{*}$) s.t. the underlaying D-mod is in the essential image of

$$\Delta^{main}_{*} : D\text{-mod}(X) \to D\text{-mod}(\mathrm{Ran}\, X)$$

Explicitly: $A \overset{ch}{\otimes} A \longrightarrow A \rightsquigarrow$

$$\mu : j_{2*} j_2^! (A \boxtimes A) \simeq (A \overset{ch}{\otimes} A)_2 \longrightarrow A_2 \simeq \Delta_{2*} A$$

Def/Prop:

Let $\mathcal{B} \in \mathrm{coAlg}(\text{D-mod}(\mathrm{Ran}\, X^{ch}))$

$$\rightsquigarrow \mathcal{B}_J \to (\mathcal{B}^{\overset{ch}{\otimes} I})_J \simeq$$

$$\bigoplus_{\alpha : J \twoheadrightarrow I} j_{(\alpha)*} j^!_{(\alpha)} \underset{I}{\boxtimes} \mathcal{B}_{J_i}$$

$$\rightsquigarrow j^!_{(\alpha)} \mathcal{B}_J \longrightarrow j^!_{(\alpha)} \underset{I}{\boxtimes} \mathcal{B}_{J_i} \qquad (*)$$

We say $\mathcal{B}$ is a factorization alg if $(*)$ is an equiv for all α.

§4 - Koszul Duality

Reminder: $\mathcal{C}$ monoidal Cat.

$\mathrm{Op}(\mathcal{C}) := \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C}^{\Sigma})$, $\mathrm{coOp}(\mathcal{C}) := \mathrm{coAlg}^{\mathrm{aug}}(\mathcal{C}^{\Sigma})$

$$\begin{tikzcd}
\mathrm{Bar} : \mathrm{Op}(\mathcal{C}) \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathrm{coOp}(\mathcal{C}) : \mathrm{coBar} \arrow[l, bend left]
\end{tikzcd}$$

$$\mathcal{O} \longmapsto \left| \cdots \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \mathbb{1}\circ\mathcal{O}\circ\mathbb{1} \rightrightarrows \mathbb{1}\circ\mathbb{1} \right|$$

$$\text{“}\mathbb{1}\otimes_{\mathcal{O}}\mathbb{1}\text{”}$$

$$\mathrm{Tot}\left(\cdots \substack{\leftarrow\\ \leftarrow\\ \leftarrow} \mathbb{1}\circ\mathcal{P}\circ\mathbb{1} \leftleftarrows \mathbb{1}\circ\mathbb{1}\right) \longmapsfrom \mathcal{P}$$

$$\begin{tikzcd}
T_{(-)} : \mathrm{Op}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Monad}(\mathcal{C}) \arrow[d] \\
\mathcal{C}^{\Sigma} \arrow[r] & \mathrm{End}(\mathcal{C}) \\
S_{(-)} : \mathrm{coOp}(\mathcal{C}) \arrow[r] \arrow[u] & \mathrm{coMonad}(\mathcal{C}) \arrow[u]
\end{tikzcd}$$

$$X \longmapsto \bigoplus_{n\geq 0}\left(\mathcal{O}(n)\otimes X^{\otimes n}\right)_{\Sigma_n}$$

$$\rightsquigarrow \quad \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) := T_{\mathcal{O}}\text{-}\mathrm{mod}(\mathcal{C})$$

$$\mathrm{coAlg}^{\mathrm{nil,d.p.}}_{\mathcal{P}}(\mathcal{C}) := S_{\mathcal{O}}\text{-}\mathrm{comod}(\mathcal{C})$$

Rem: $\mathrm{coAlg}_{\mathcal{P}}(\mathcal{C})$ will corresp to

$$X \longmapsto \underbrace{\boxed{\prod_{n\geq 0}}}_{\text{nil}}\left(\mathcal{O}(n)\otimes X^{\otimes n}\right)^{\underbrace{\boxed{\Sigma_n}}_{\text{d.p.}}}$$

Prop: $\mathrm{Bar}(\mathrm{Lie}) \simeq \mathrm{coComm}[-1]$

Idea: $0 \to \mathrm{Lie} \to \mathrm{Assoc} \to \mathrm{Comm} \to 0$

So we want a map

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \longrightarrow \mathrm{coAlg}_{\mathrm{Bar}(\mathcal{O})}(\mathcal{C})$$

Let
$$\begin{tikzcd}\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r, bend left, "\epsilon_*"] \arrow[r, phantom, "T"] & \mathrm{Alg}_{\mathbb{1}}(\mathcal{C}) \simeq \mathcal{C} \arrow[l, bend left, "\epsilon^*"]\end{tikzcd}$$

be the pull push along the augmentation.

Lemma: $\epsilon^*\epsilon_* \simeq S_{\mathrm{Bar}(\mathcal{O})}$

Idea: $S_{\mathrm{Bar}(\mathcal{O})} \simeq S_{\mathbb{1}\otimes_{\mathcal{O}}\mathbb{1}}(x)$

$$\simeq \left| \cdots \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow}} \mathbb{1}\otimes\mathcal{O}\otimes\mathbb{1} \underset{id\otimes\epsilon}{\overset{\epsilon\otimes id}{\rightrightarrows}} \mathbb{1}\otimes\mathbb{1} \right| (x)$$

$$\simeq \left| \cdots \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow}} id\circ T_{\mathcal{O}}\circ id(X) \underset{id\circ\epsilon_X}{\overset{\epsilon_X\circ id}{\rightrightarrows}} id\circ id(X) \right|$$

action commutes
w/ geo. realization

$$\simeq \left| \cdots \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow}} T_{\mathcal{O}}\epsilon_*X \rightrightarrows \epsilon_*X \right|$$

$$\simeq \mathbb{1}\otimes_{\mathcal{O}}\epsilon_*X \simeq \epsilon^*\epsilon_*X$$

Cor:

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{G}}(\mathcal{C}) \arrow[rr] \arrow[dr, dashed] \arrow[ddr, bend right, "KD_{\mathcal{G}\to\mathcal{G}^\vee} :=" '] & & \mathcal{C} \\
& \epsilon^* \epsilon_*\text{-comod}(\mathcal{C}) \arrow[ur] \arrow[d, phantom, sloped, "\simeq"] & \\
& \mathrm{coAlg}^{nil,\,d.p.}_{\mathrm{Bar}(\mathcal{G})}(\mathcal{C}) &
\end{tikzcd}$$

Conj:

$$\begin{tikzcd}
\mathrm{Alg}^{nil}_{\mathcal{G}}(\mathcal{C}) \arrow[r, hook] \arrow[rr, bend right, "\sim" '] & \mathrm{Alg}_{\mathcal{G}}(\mathcal{C}) \arrow[r] & \mathrm{coAlg}^{nil,\,d.p.}_{\mathrm{Bar}(\mathcal{G})}(\mathcal{G})
\end{tikzcd}$$

(FALSE in general!)

Will be true in our case.

Upshot: $D\text{-mod}(\mathrm{Ran}\,X) \simeq \mathrm{colim}\, D\text{-mod}(\mathrm{Ran}^{\leq n} X)$

$\rightsquigarrow$ pro nilpotent + char 0 $\rightsquigarrow$

$\mathrm{Alg}^{nil} \simeq \mathrm{Alg}$, $\mathrm{coAlg}^{nil, d.p.} \simeq \mathrm{coAlg}$ $\rightsquigarrow$

Thm: $KD_{Lie \to Comm} \simeq CE$ induces

$$\mathrm{Lie}(D\text{-mod}(\mathrm{Ran}X^{ch})) \xrightarrow{\sim} \mathrm{coCAlg}(D\text{-mod}(\mathrm{Ran}X^{ch}))$$

Thm: Restricts to an equiv

$$\mathrm{ChAlg}(X) \xrightarrow{\sim} \mathrm{FactAlg}(X)$$

Idea: $A \in \mathrm{ChAlg}(X)$,

$\underline{CE(A)} \simeq \mathrm{Sym}(A[1]) \simeq$

$\rightsquigarrow$ underlaying D-module

$$\xrightarrow{\lim} \mathrm{Sym}^{\leq n}(\mathcal{A}[1])$$

$$\rightsquigarrow \mathrm{gr}^{I} CE(\mathcal{A}) \simeq \mathrm{Sym}^{I}(\mathcal{A}[1]) \simeq \left(\mathcal{A}^{\overset{ch}{\otimes} I}\right)_{\Sigma_I}$$

$$\mathrm{gr}^{I} CE(\mathcal{A})_J \simeq \Big(\bigoplus_{\alpha : J \twoheadrightarrow I} j_{(\alpha)*} j^{!}_{(\alpha)} \underset{I}{\boxtimes} \mathcal{A}_{J_i} \Big)_{\Sigma_I}$$

The factorization property is then equiv to

$$j_I^{!} CE(\mathcal{A})_I \xrightarrow{\sim} j_I^{!} \mathcal{A}[1]_1^{\boxtimes I}$$

$$\simeq j_I^{!} \mathrm{gr}^{I} CE(\mathcal{A})_I$$

i.e. $j_I^{!} \mathrm{gr}^{I} CE(\mathcal{A})_J = 0 \quad |J| \neq |I|$.

This happens precisely if $\mathcal{A}_J$ is supported on a diagonal, so that it will be annihilated

by $y_I^!$, which in turn is equiv.
to $A \simeq \Delta_*^{main} A_1$.